Multiderivations of Coxeter arrangements

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Dedicated to Professor Tatsuo Suwa for his sixtieth birthday

Abstract

Let V be an ℓ-dimensional Euclidean space. Let $G \subset O(V)$ be a finite irreducible orthogonal reflection group. Let $\mathcal{A}$ be the corresponding Coxeter arrangement. Let S be the algebra of polynomial functions on V. For $H \in \mathcal{A}$ choose $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$. For each nonnegative integer m, define the derivation module $\mathsf{D}^{(m)}(\mathcal{A}) = \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in S\alpha_H^m\}$. The module is known to be a free S-module of rank ℓ by K. Saito (1975) for $m = 1$ and L. Solomon-H. Terao (1998) for $m = 2$. The main result of this paper is that this is the case for all m. Moreover we explicitly construct a basis for $\mathsf{D}^{(m)}(\mathcal{A})$. Their degrees are all equal to $mh/2$ (when m is even) or are equal to $((m-1)h/2) + m_i(1 \leq i \leq \ell)$ (when m is odd). Here $m_1 \leq \cdots \leq m_\ell$ are the exponents of G and $h = m_\ell + 1$ is the Coxeter number. The construction heavily uses the primitive derivation D which plays a central role in the theory of flat generators by K. Saito (or equivalently the Frobenius manifold structure for the orbit space of G.) Some new results concerning the primitive derivation D are obtained in the course of proof of the main result.

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1   Introduction

Let V be a Euclidean space of dimension ℓ over $\mathbb{R}$. Let $\mathcal{A}$ be a central arrangement of hyperplanes in V : $\mathcal{A}$ is a finite collection of one-codimensional subspaces of V. Let S be the algebra of polynomial functions on V. The algebra S is naturally graded by $S = \oplus_{q \geq 0} S_q$ where S_q is the space of homogeneous

polynomials of degree q. Thus $S_1 = V^*$ is the dual space of V. Let Der_S be the S-module of $\mathbb{R}$-derivations of S. We say that $\theta \in \mathrm{Der}_S$ is homogeneous of degree q if $\theta(S_1) \subseteq S_q$. Choose for each hyperplane $H \in \mathcal{A}$ a linear form $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$. Let

(1) $\qquad \mathsf{D}^{(m)}(\mathcal{A}) = \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in S\alpha_H^m \text{ for any } H \in \mathcal{A}\}$

for each nonnegative integer m. Elements of $\mathsf{D}^{(m)}(\mathcal{A})$ are called m-**derivations** which were introduced by G. Ziegler [Zie]. Then one has a sequence of inclusions

$$\mathsf{D}^{(1)}(\mathcal{A}) \supset \mathsf{D}^{(2)}(\mathcal{A}) \supset \cdots .$$

The arrangement $\mathcal{A}$ is called **free** [OrT, Chapter 4] when $\mathsf{D}^{(1)}(\mathcal{A}) = \mathsf{D}(\mathcal{A})$ is a free S-module.

From now on we assume that $\mathcal{A}$ is a Coxeter arrangement : $\mathcal{A}$ is the set of reflecting hyperplanes of a finite irreducible subgroup G of $O(V)$ generated by orthogonal reflections. Then G naturally acts on V^* and S. Let S^G be the algebra of G-invariant polynomials. Then it is classically known [Bou, V.5.3, Theorem 3] that there exist algebraically independent homogeneous polynomials $f_1, \cdots, f_\ell \in S^G$ with $\deg f_1 \leq \cdots \leq \deg f_\ell$, which are called **basic invariants**, such that $S^G = \mathbb{R}[f_1, \cdots, f_\ell]$. Write $\mathbf{f} = (f_1, \cdots, f_\ell)$. Let $x_1, \cdots, x_\ell$ be a basis for V^* and ∂_i be partial derivation with respect to x_i: $\partial_i(x_j) = \delta_{ij}$ (Kronecker's delta). Let K be the quotient field of S. The **primitive derivation** $D \in \mathrm{Der}_K$ is characterized by

$$Df_i = \begin{cases} 1 & \text{for } i = \ell, \\ 0 & \text{otherwise.} \end{cases}$$

The derivation D is, up to a constant multiple, independent of choice of basic invariants $f_1, \cdots, f_\ell$ [Sai2, 2.2], [SYS, (1.6)]. For $\mathbf{g} = (g_1, \cdots, g_\ell) \in K^\ell$ let $J(\mathbf{g})$ be the Jacobian matrix with respect to $\mathbf{x} = (x_1, \cdots, x_\ell)$: $J(\mathbf{g})_{ij} = \partial_i g_j (1 \leq i, j \leq \ell)$. Let $D^k \mathbf{x} = (D^k x_1, \cdots, D^k x_\ell)$, where $D^k = D \circ D \circ \cdots \circ D$ (composition of k times). Define $\Gamma \in GL_\ell(\mathbb{R})$ by $\Gamma_{ij} = (x_i, x_j)$, where $(\ ,\ )$ stands for the G-invariant positive definite symmetric bilinear form V^*. The main results of this paper are:

Theorem 1.1. *For any nonnegative integer k, the $\ell \times \ell$-matrix $J(D^k \mathbf{x})$ is invertible. For each nonnegative integer m define an $\ell \times \ell$-matrix P_m by*

$$P_m = \begin{cases} \Gamma J(D^k \mathbf{x})^{-1} & (\text{if } m = 2k \text{ is even}), \\ \Gamma J(D^k \mathbf{x})^{-1} J(\mathbf{f}) & (\text{if } m = 2k+1 \text{ is odd}). \end{cases}$$

Define $\xi_1, \ldots, \xi_l \in \mathrm{Der}_S$ by $\xi_j = \sum_{i=1}^l p_{ij} \partial_i$ for $1 \leq j \leq l$, where p_{ij} is the (i, j)-entry of P_m. Then $\xi_i \in \mathsf{D}^{(m)}(\mathcal{A})$, and $\mathsf{D}^{(m)}(\mathcal{A})$ is a free S-module with basis $\xi_1, \ldots, \xi_l$. Each ξ_i is homogeneous of degree kh (when $m = 2k$) or of degree $kh + m_i$ (when $m = 2k+1$) for $i = 1, \ldots, \ell$, where h is the Coxeter number and $m_1, \cdots, m_\ell$ are the exponents of G.

Let $Q := \prod_{H \in \mathcal{A}} \alpha_H \in S$.

Theorem 1.2. *Let k be any nonnegative integer. Then $\det J(D^k \mathbf{x})$ is a nonzero multiple of Q^{-2k}.*

Corollary 1.3. *Let k be any nonnegative integer. Then the ℓ rational functions $D^k x_1, D^k x_2, \cdots, D^k x_\ell$ are algebraically independent.*

Remark 1.4. For $m = 1$, $P_1 = \Gamma J(\mathbf{f})$, $\mathsf{D}^{(1)}(\mathcal{A}) = \mathsf{D}(\mathcal{A})$ and Theorem 1.1 was first proved by K. Saito [Sai1, Theorem], [Ter, Theorem2]. In other words, the Coxeter arrangement is a free arrangement. For $m = 2$, Theorem 1.1 was proved in [SoT, Theorem 1.4]. Theorem 1.2 and Corollary 1.3 for $k = 1$ were also proved in [SoT, Corollary 3.32]. Corollary 1.3 for $k = 1$ had been conjectured by L. Solomon in [Sol2] in 1964.

Remark 1.5. The definition (1) of $\mathsf{D}^{(m)}(\mathcal{A})$ is due to G. Ziegler [Zie, Definition 4] who developed the theory of multiarrangements.

Remark 1.6. The original motivation to study the module $\mathsf{D}^{(m)}(\mathcal{A})$ came from the study of the extended Shi arrangements and the extended Catalan arrangements. Suppose that G is a Weyl group. Choose a crystallographic root system in V^* and choose the linear forms α_H so that $\pm \alpha_H$ is a root for each $H \in \mathcal{A}$. Let $n = |\mathcal{A}|$ and $\alpha_1, \ldots, \alpha_n \in V^*$ be a system of positive roots. Let $H_{i,p}$ be the affine hyperplane defined by $\alpha_i = p$ for $p \in \mathbb{Z}$ and $1 \leq i \leq n$. Let $k \geq 1$. Define the extended Shi arrangement[Shi1][Shi2][Ath2]

$$\mathcal{A}^{[-k+1,k]} := \{H_{i,p} \mid 1 \leq i \leq n, -k+1 \leq p \leq k\},$$

and the extended Catalan arrangement

$$\mathcal{A}^{[-k,k]} := \{H_{i,p} \mid 1 \leq i \leq n, -k \leq p \leq k\}.$$

For these arrangements the characteristic polynomials [OrT, p.43] are known [Pos][PSt] [Ath1, §7.1-7.2][Ath2] to decompose nicely as:

$$
\begin{aligned}
\chi(\mathcal{A}^{[-k+1,k]}, t) &= (t - kh)^\ell, \\
\chi(\mathcal{A}^{[-k,k]}, t) &= \prod_{i=1}^{\ell} (t - kh - m_i),
\end{aligned}
$$

when G is of type $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$. This result is obtained by direct calculation using the classification of Weyl groups. The same formulas are conjectured to be true for the other Weyl groups. We are led to study the module $\mathsf{D}^{(m)}(\mathcal{A})$ to understand the meaning of these roots $kh, \ldots, kh$ (for $\mathcal{A}^{[-k+1,k]}$) and $kh + m_1, \ldots, kh + m_\ell$ (for $\mathcal{A}^{[-k,k]}$) for the characteristic polynomials. Let $\mathcal{A}$ be either an extended Shi arrangement or an extended Catalan arrangement. A conjecture due to Edelman and Reiner [EdR, Conjecture 3.3] states that the cone [OrT,

p.14] $\mathbf{c}\mathcal{A}$ is a free arrangement with exponents $\{1, kh, \ldots, kh\}$ (for $\mathcal{A}^{[-k+1,k]}$) or $\{1, kh + m_1, \ldots, kh + m_\ell\}$ (for $\mathcal{A}^{[-k,k]}$) [OrT, Definition 4.15, Definition 4.25]; the module $D(\mathbf{c}\mathcal{A})$ is a free module over $\mathbf{R}[x_0, \ldots, x_l]$. Athanasiadis [Ath1] verified this conjecture for type A_l. If the conjecture is true, then, by Ziegler's criterion, which will be stated as Theorem 3.1, we may conclude that $\mathsf{D}^{(2k)}(\mathcal{A})$ is a free module with generators of degrees $\{kh, kh, \ldots, kh\}$ (for $\mathcal{A}^{[-k+1,k]}$) and that $\mathsf{D}^{(2k+1)}(\mathcal{A})$ is a free module with generators of degrees $\{kh + m_1, kh + m_2, \ldots, kh + m_\ell\}$ (for $\mathcal{A}^{[-k,k]}$) which is true by our main result, Theorem 1.1. So Theorem 1.1 may be regarded as a piece of evidence supporting the conjecture.

2 Notation and preliminaries

In this Section we fix more notation and state preliminary facts which will be used later. A finite irreducible orthogonal reflection group G acts on an ℓ-dimensional Euclidean space V. Then G acts on the dual space V^* contragradiently. In other words,

$$\langle gx, v \rangle = \langle x, g^{-1}v \rangle$$

for $x \in V^*$, $v \in V$ and $g \in G$ when $\langle \, , \, \rangle$ denotes the natural pairing of V and V^*. The symmetric algebra $S = S(V^*)$ of V^* over $\mathbb{R}$ can be identified as the $\mathbb{R}$-algebra of all polynomial functions on V. The contragradient action of G on V^* extends to the action on S by $(gf)(v) = f(g^{-1}v)$ for $f \in S$, $g \in G$ and $v \in V$. Each element of the graded S-module Der_S is an $\mathbb{R}$-derivation of S, i.e., an $\mathbb{R}$-linear map $\theta : S \to S$ satisfying the Leibniz rule: $\theta(f_1 f_2) = f_1\theta(f_2) + f_2\theta(f_1)$ for $f_1, f_2 \in S$. The action of G on Der_S is defined by $(g\theta)(f) = g(\theta(g^{-1}f))$ for $g \in G$, $\theta \in \mathrm{Der}_S$ and $f \in S$. The group G naturally acts on the quotient field K of S by

$$g(f_1/f_2) = (gf_1)/(gf_2)$$

for $f_1 \in S$, $f_2 \in S \setminus \{0\}$ and $g \in G$. When $f_1 \in S_p$ and $f_2 \in S_q \setminus \{0\}$ define $\deg(f_1/f_2) = p - q$. For a matrix A of any size with entries in K, A_{ij} is the (i, j)-entry of A. Let $A^\top$ denote the transpose of A. Let $g \in G$. Define the matrix $g[A]$ by

$$g[A]_{ij} = g(A_{ij}).$$

Choose a basis $x_1, \ldots, x_\ell$ for V^* and $\mathbb{R}$-derivations $\partial_1, \ldots, \partial_\ell \in \mathrm{Der}_S$ such that

$$\partial_i(x_j) = \delta_{ij}$$

for $1 \leq i, j \leq \ell$. Then $\deg \partial_i = 0$ and Der_S is a free S-module with basis $\partial_1, \ldots, \partial_\ell$. The invertible matrix $\rho(g) \in GL_\ell(\mathbb{R})$ is defined by

$$(2) \qquad\qquad gx_j = \sum_{i=1}^{\ell} \rho(g)_{ij} x_i,$$

or $g[\mathbf{x}] = \mathbf{x}\rho(g)$. Then

$$g\partial_j = \sum_{i=1}^{\ell} \rho(g)_{ji}^{-1}\partial_i.$$

One can symbolically express this as

(3) $$(g\partial_1, \ldots, g\partial_\ell) = (\partial_1, \ldots, \partial_\ell)(\rho(g)^\top)^{-1}.$$

Let $(\ ,\ )$ denote the positive definite bilinear form on V^* inverse to the form on V. Recall the $\ell \times \ell$-matrix $\Gamma = [(x_i, x_j)]$. For $g \in G$, we have

(4) $$\Gamma = \rho(g)^\top \Gamma \rho(g).$$

Choose basic invariants $f_1, \ldots, f_\ell \in S^G$ with $\deg f_1 \leq \cdots \leq \deg f_\ell$. Let $m_i = \deg f_i - 1$ for $i = 1, \ldots, \ell$. Then the integers $m_1, \ldots, m_\ell$ are the **exponents** of G. Since G is irreducible, one has [Bou, V.6.2]

$$1 = m_1 < m_2 \leq \cdots \leq m_{\ell-1} < m_\ell.$$

Let h denote the Coxeter number: $h = m_\ell + 1$ [Bou, V.6.2, Theorem 1]. Then we have the duality of exponents [Bou, V.6.2]:

$$m_i + m_j = h \quad (\text{if } i + j = \ell + 1).$$

Write $\mathbf{f} = (f_1, \ldots, f_\ell)$. Then, for $g \in G$, we have

(5) $$g[J(\mathbf{f})] = \rho(g)^{-1}J(\mathbf{f}).$$

By [Bou, V.5.5, Proposition 6 (ii)] we have

(6) $$\det J(\mathbf{f}) \doteq Q.$$

Here and elsewhere $\doteq$ stands for equality up to a nonzero constant multiple.

Let Der_K be the set of $\mathbb{R}$-derivations of K. The primitive derivation D is defined to be the unique derivation satisfying $Df_i = 0$ $(1 \leq i \leq \ell - 1)$ and $Df_\ell = 1$. Since

$$D = \sum_{j=1}^{\ell}(Dx_j)\partial_j$$

we have

$$\delta_{k\ell} = Df_k = \sum_{j=1}^{\ell}(Dx_j)(\partial_j f_k).$$

Thus $D\mathbf{x} = (Dx_1, \ldots, Dx_\ell)$ is equal to the bottom row of the matrix $J(\mathbf{f})^{-1}$:

(7) $$J(\mathbf{f})_{\ell j}^{-1} = Dx_j \quad (j = 1, \ldots, \ell).$$

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By (6) we know that $Q(Dx_j) \in S$ for $1 \leq j \leq \ell$. For $k \geq 0$, an $\mathbb{R}$-linear map $D^k : K \to K$ is defined by composing D k times. Agree that D^0 is the identity map. Then $D^k x_j$ has no pole outside the reflecting hyperplanes. Since $\deg(Dx_j) = -m_\ell = -h+1$ unless $Dx_j = 0$, one has

$$(8) \qquad \qquad \deg(\partial_i D^k x_j) \quad = \quad -kh$$

unless $\partial_i D^k x_j = 0$. For $D^k \mathbf{x} = (D^k x_1, \ldots, D^k x_\ell)$ and $g \in G$, we obtain $g[D^k \mathbf{x}] = (D^k \mathbf{x})\rho(g)$ and

$$(9) \qquad \qquad g[J(D^k \mathbf{x})] \quad = \quad \rho(g)^{-1} J(D^k \mathbf{x})\rho(g).$$

Remark 2.1. The primitive derivation D is, up to a nonzero constant multiple, independent of choice of basic invariants $f_1, \ldots, f_\ell$ [Sai2, 2.2], [SYS, (1.6)]. The derivation D is called the primitive vector field by K. Saito in [Sai5] and plays a central role in his theory of flat generators and primitive integrals [Sai2] [Sai5]. (See Remark 3.7 (ii).)

Suppose that another basis $x'_1, \ldots, x'_\ell$ for V^* is connected with $x_1, \ldots, x_\ell$ through an invertible matrix $M \in GL_\ell(\mathbb{R})$:

$$(10) \qquad \qquad x'_j = \sum_{i=1}^{\ell} M_{ij} x_i,$$

or $\mathbf{x}' = \mathbf{x}M$. The new objects, which are defined using the new basis $x'_1, \ldots, x'_\ell$, will be denoted by ∂'_i, Γ', J' etc. Then one can symbolically express

$$(11) \qquad \qquad (\partial'_1, \ldots, \partial'_\ell) = (\partial_1, \ldots, \partial_\ell)(M^\top)^{-1}$$

We also have

$$(12) \qquad \qquad \Gamma' \quad = \quad M^\top \Gamma M$$
$$(13) \qquad \qquad J'(\mathbf{f}) \quad = \quad M^{-1} J(\mathbf{f})$$
$$(14) \qquad \qquad J'(D^k \mathbf{x}') \quad = \quad M^{-1} J(D^k \mathbf{x})M.$$

3   Proof of Theorems

In this section we prove Theorems 1.1 and 1.2. First we split the assertions into halves:

$(E)_k$: *The determinant of $\ell \times \ell$-matrix $J(D^k \mathbf{x})$ is a nonzero multiple of Q^{-2k} and define an $\ell \times \ell$-matrix P_{2k} by*

$$P_{2k} = \Gamma J(D^k \mathbf{x})^{-1}.$$

Define $\xi_1, \ldots, \xi_l \in \mathrm{Der}_S$ by $\xi_j = \sum_{i=1}^{l} p_{ij} \partial_i$ for $1 \leq j \leq l$, where p_{ij} is the (i,j)-entry of P_{2k}. Then $\xi_j \in \mathsf{D}^{(2k)}(\mathcal{A})$, and $\mathsf{D}^{(2k)}(\mathcal{A})$ is a free S-module with

basis $\xi_1, \dots, \xi_l$. Each ξ_j is homogeneous of degree kh $(1 \le j \le \ell)$ where h is the Coxeter number.

$(O)_k$: *The determinant of $\ell \times \ell$-matrix $J(D^k \mathbf{x})$ is a nonzero multiple of Q^{-2k} and define an $\ell \times \ell$-matrix P_{2k+1} by*

$$P_{2k+1} = \Gamma J(D^k \mathbf{x})^{-1} J(\mathbf{f}).$$

Define $\theta_1, \dots, \theta_l \in \mathrm{Der}_S$ by $\theta_j = \sum_{i=1}^{l} p_{ij} \partial_i$ for $1 \le j \le l$, where p_{ij} is the (i,j)-entry of P_{2k+1}. Then $\theta_j \in \mathsf{D}^{(2k+1)}(\mathcal{A})$, and $\mathsf{D}^{(2k+1)}(\mathcal{A})$ is a free S-module with basis $\theta_1, \dots, \theta_l$. Each θ_j is homogeneous of degree $kh + m_j$ $(1 \le j \le \ell)$ where h is the Coxeter number and $m_1, \cdots, m_\ell$ are the exponents of G.

The following theorem is a special case of a theorem due to G. Ziegler [Zie, Theorem 11]. It generalizes Saito's criterion [Sai3, p.270], [OrT, Theorem 4.19].

Theorem 3.1. (Ziegler's criterion) *Let $\theta_1, \dots, \theta_\ell \in D^{(m)}(\mathcal{A})$. Then they form a basis for $D^{(m)}(\mathcal{A})$ if and only if $\det [\theta_j(x_i)]_{ij} \doteq Q^m$.*

Lemma 3.2. *For any nonnegative integer k, the assertion $(O)_k$ follows from $(E)_k$.*

Proof. One can symbolically express

$$(\theta_1, \dots, \theta_\ell) = (\partial_1, \dots, \partial_\ell) P_{2k+1}.$$

Let $\rho(g) \in GL_\ell(\mathbb{R})$ satisfy (2):

$$g x_j = \sum_{i=1}^{\ell} \rho(g)_{ij} x_i.$$

We have

$$
\begin{aligned}
g[P_{2k+1}] &= g[\Gamma J(D^k \mathbf{x})^{-1} J(\mathbf{f})] \\
&= \rho(g)^\top \Gamma \rho(g) \rho(g)^{-1} J(D^k \mathbf{x})^{-1} \rho(g) \rho(g)^{-1} J(\mathbf{f}) = \rho(g)^\top P_{2k+1}
\end{aligned}
$$

by (4), (9) and (5). This implies that each θ_j is G-invariant because of (3):

$$(g\partial_1, \dots, g\partial_\ell) = (\partial_1, \dots, \partial_\ell)(\rho(g)^\top)^{-1}$$

Let $H \in \mathcal{A}$. Since

$$\theta_j = \sum_{i=1}^{\ell} (\partial_i f_j) \xi_i,$$

each θ_j belongs to $\mathsf{D}^{(2k)}(\mathcal{A})$. Write $\theta_j(\alpha_H) = \alpha_H^{2k} p_j(\mathbf{x})$ for some $p_j(\mathbf{x}) \in S$. Let s_H be the orthogonal reflection about H. Then

$$
\begin{aligned}
\alpha_H^{2k} s_H[p_j(\mathbf{x})] &= (-\alpha_H)^{2k} s_H[p_j(\mathbf{x})] = s_H[\alpha_H^{2k} p_j(\mathbf{x})] = s_H[\theta_j(\alpha_H)] \\
&= \theta_j(s_H \alpha_H) = \theta_j(-\alpha_H) = -\theta_j(\alpha_H) = -\alpha_H^{2k} p_j(\mathbf{x}).
\end{aligned}
$$

Therefore one has

$$s_H[p_j(\mathbf{x})] = -p_j(\mathbf{x}).$$

Hence $p_j(\mathbf{x})$ is divisible by α_H and thus $\theta_j(\alpha_H) = \alpha_H^{2k} p_j(\mathbf{x}) \in S\alpha_H^{2k+1}$. This implies $\theta_j \in \mathsf{D}^{(2k+1)}(\mathcal{A})$ because $H \in \mathcal{A}$ was arbitrary. Recall Ziegler's criterion 3.1. Since $\det P_{2k+1} = \det(P_{2k}J(\mathbf{f})) \doteq Q^{2k+1}$, we can conclude that $\theta_1, \ldots, \theta_\ell \in \mathsf{D}^{2k+1}(\mathcal{A})$ form a basis for the S-module $\mathsf{D}^{2k+1}(\mathcal{A})$. We also have

$$\deg \theta_j = \deg \xi_i + \deg(\partial_i f_j) = kh + m_j.$$

$\square$

Lemma 3.3. *Let k be any nonnegative integer. Assume that $(E)_k$ holds true. Define an $\ell \times \ell$-matrix $B^{(k+1)}$ with entries in K by*

(15) $$B^{(k+1)} := -J(\mathbf{f})^\top \Gamma J(D^{k+1}\mathbf{x}) J(D^k\mathbf{x})^{-1} J(\mathbf{f}).$$

Then
  (i) $B_{ij}^{(k+1)} \in S^G$,
  (ii) $\deg B_{ij}^{(k+1)} = m_i + m_j - h \quad (1 \le i, j \le \ell)$.

Proof. (i): Let $g \in G$. By (5), (9) and (4), one has

$$
\begin{aligned}
&g[B^{(k+1)}] \\
=\ & g[-J(\mathbf{f})^\top \Gamma J(D^{k+1}\mathbf{x}) J(D^k\mathbf{x})^{-1} J(\mathbf{f})] \\
=\ & -J(\mathbf{f})^\top (\rho(g)^\top)^{-1} \rho(g)^\top \Gamma \rho(g) \rho(g)^{-1} J(D^{k+1}\mathbf{x}) \rho(g) \rho(g)^{-1} J(D^k\mathbf{x})^{-1} \rho(g) \rho(g)^{-1} J(\mathbf{f}) \\
=\ & B^{(k+1)}.
\end{aligned}
$$

Thus the matrix $B^{(k+1)}$ is G-invariant.

Next we will show that $B^{(k+1)}$ is independent of choice of the basis $x_1, \cdots, x_\ell$ for V^*. Suppose that a new basis $x_1', \cdots, x_\ell'$ for V^* is connected with the old basis $x_1, \cdots, x_\ell$ through an invertible matrix $M \in GL_\ell(\mathbb{R})$ as (10):

$$x_j' = \sum_{i=1}^\ell M_{ij} x_i.$$

The new objects, which are defined using the new basis, will be denoted by Γ', J' etc. By (13), (12) and (14), one has

$$
\begin{aligned}
B^{(k+1)'} &= -J'(\mathbf{f})^\top \Gamma' J'(D^{k+1}\mathbf{x}') J'(D^k\mathbf{x}')^{-1} J'(\mathbf{f}) \\
&= -J(\mathbf{f})^\top (M^{-1})^\top M^\top \Gamma M M^{-1} J(D^{k+1}\mathbf{x}) M M^{-1} J(D^k\mathbf{x})^{-1} M M^{-1} J(\mathbf{f}) \\
&= B^{(k+1)}
\end{aligned}
$$

Therefore $B^{(k+1)}$ is independent of choice of $x_1, \cdots, x_\ell$.

Let $H \in \mathcal{A}$. Then we may choose an orthonormal basis $x_1 = \alpha_H, x_2, \cdots, x_\ell$ for V^* without affecting $B^{(k+1)}$. Then Γ is the identity matrix. By Lemma 3.2, the assertion $(O)_k$ holds. Thus each entry of the first row of $J(D^k\mathbf{x})^{-1}J(\mathbf{f})$ is divisible by x_1^{2k+1}. Thus, outside the first column, each entry of $\mathrm{adj}(J(D^k\mathbf{x})^{-1}J(\mathbf{f}))$ is divisible by x_1^{2k+1}. Since $\det J(D^k\mathbf{x})^{-1}J(\mathbf{f}) \doteq Q^{2k+1}$ each entry of

$$(J(D^k\mathbf{x})^{-1}J(\mathbf{f}))^{-1} = J(\mathbf{f})^{-1}J(D^k\mathbf{x}),$$

outside the first column, has no pole along $x_1 = 0$. The (ℓ, j)-entry of $J(\mathbf{f})^{-1}J(D^k\mathbf{x})$ is equal to

$$\sum_{i=1}^{\ell} J(\mathbf{f})_{\ell i}^{-1}\partial_i(D^k x_j) = \sum_{i=1}^{\ell}(Dx_i)\partial_i(D^k x_j) = D^{k+1}x_j$$

by (7). Thus each entry, except the first, of $D^{k+1}\mathbf{x}$ has no pole along $x_1 = 0$. Note that the first entry of $D^{k+1}\mathbf{x}$ has pole of order at most $2k+1$ along $x_1 = 0$. It follows that each entry of $J(D^{k+1}\mathbf{x})$, outside the first column, has no pole along $x_1 = 0$. Each entry of the first column of $J(D^{k+1}\mathbf{x})$ has pole of order at most $2k+2$ along $x_1 = 0$. Since each entry of the first row of $J(D^k\mathbf{x})^{-1}J(\mathbf{f})$ is divisible by x_1^{2k+1}, each entry of the product $J(D^{k+1}\mathbf{x})J(D^k\mathbf{x})^{-1}J(\mathbf{f})$ has pole of order at most one along $\alpha_H = x_1 = 0$. So

$$\alpha_H B^{(k+1)} = -\alpha_H J(\mathbf{f})^\top \Gamma J(D^{k+1}\mathbf{x})J(D^k\mathbf{x})^{-1}J(\mathbf{f})$$

has no pole along H. Since $H \in \mathcal{A}$ was arbitrarily chosen one can conclude that $QB_{ij}^{(k+1)} \in S$. For any $g \in G$, we have

$$g[QB_{ij}^{(k+1)}] = \det(g)^{-1}QB_{ij}^{(k+1)}.$$

Thus $QB_{ij}^{(k+1)}$ is an anti-invariant [Bou, V. 5,5, Proposition 6 (iv)] and hence lies in QS^G. Therefore $B_{ij}^{(k+1)} \in S^G$. This proves (i).

(ii): By (8), we have

$$
\begin{aligned}
\deg B_{ij}^{(k+1)} &= \deg((\partial_1 f_i, \ldots, \partial_\ell f_i)\Gamma J(D^{k+1}\mathbf{x})J(D^k\mathbf{x})^{-1}(\partial_1 f_j, \ldots, \partial_\ell f_j)^\top) \\
&= m_i - (k+1)h + kh + m_j = m_i + m_j - h.
\end{aligned}
$$

$\square$

Lemma 3.4. *For* $\mathbf{g} = (g_1, \ldots, g_\ell) \in K^\ell$, *we have*
   (i) $J(D\mathbf{g}) = J(D\mathbf{x})J(\mathbf{g}) + D[J(\mathbf{g})]$,
   (ii) $D[J(\mathbf{g})^{-1}] = -J(\mathbf{g})^{-1}D[J(\mathbf{g})]J(\mathbf{g})^{-1}$,
   (iii) $D[J(\mathbf{f})] = -J(D\mathbf{x})J(\mathbf{f})$,
   (iv) $D[J(\mathbf{g})^{-1}J(\mathbf{f})] = -J(\mathbf{g})^{-1}J(D\mathbf{g})J(\mathbf{g})^{-1}J(\mathbf{f})$.

Proof. (i) The (i,j)-entry of $J(D\mathbf{g})$ is equal to:

$$\partial_i(Dg_j) \quad = \quad \partial_i(\sum_{p=1}^{\ell}(Dx_p)(\partial_p g_j)) = \sum_{p=1}^{\ell}(\partial_i Dx_p)(\partial_p g_j) + \sum_{p=1}^{\ell}(Dx_p)(\partial_i \partial_p g_j)$$

$$= \quad \sum_{p=1}^{\ell}(\partial_i Dx_p)(\partial_p g_j) + D(\partial_i g_j),$$

which is equal to the (i,j)-entry of $J(D\mathbf{x})J(\mathbf{g}) + D[J(\mathbf{g})]$.

(ii) follows from:

$$0 = D[J(\mathbf{g})J(\mathbf{g})^{-1}] = D[J(\mathbf{g})]J(\mathbf{g})^{-1} + J(\mathbf{g})D[J(\mathbf{g})^{-1}].$$

(iii) Let $\mathbf{g} = \mathbf{f}$ in (i) to get $J(D\mathbf{f}) = J(D\mathbf{x})J(\mathbf{f}) + D[J(\mathbf{f})]$. Apply

$$J(D\mathbf{f}) = J(Df_1, \ldots, Df_\ell) = J(0, \ldots, 0, 1) = 0.$$

(iv) By (i), (ii), and (iii), one has

$$D[J(\mathbf{g})^{-1}J(\mathbf{f})] = -J(\mathbf{g})^{-1}D[J(\mathbf{g})]J(\mathbf{g})^{-1}J(\mathbf{f}) + J(\mathbf{g})^{-1}D[J(\mathbf{f})]$$

$$= \quad -J(\mathbf{g})^{-1}\left(J(D\mathbf{g}) - J(D\mathbf{x})J(\mathbf{g})\right)J(\mathbf{g})^{-1}J(\mathbf{f}) - J(\mathbf{g})^{-1}J(D\mathbf{x})J(\mathbf{f})$$

$$= \quad -J(\mathbf{g})^{-1}J(D\mathbf{g})J(\mathbf{g})^{-1}J(\mathbf{f}).$$

$\square$

Lemma 3.5. *Let k be any positive integer. Assume that $(E)_{k-1}$ and $(E)_k$ both hold true. Define $B^{(k)}$ and $B^{(k+1)}$ as (15). Then*

$$B^{(k+1)} - B^{(k)} = B^{(1)} + (B^{(1)})^{\top}.$$

In particular, $B^{(k+1)} - B^{(k)}$ is a symmetric matrix.

Proof. Since $B_{ij}^{(k)} \in S^G$ and

$$\deg B_{ij}^{(k)} = m_i + m_j - h \leq (h-1) + (h-1) - h = h - 2 < \deg f_\ell$$

by Lemma 3.3, each entry of $B^{(k)}$ lies in $\mathbb{R}[f_1, \ldots, f_{\ell-1}]$. Thus

$$0 \quad = \quad D[B^{(k)}] = D[-J(\mathbf{f})^{\top}\Gamma J(D^k\mathbf{x})J(D^{k-1}\mathbf{x})^{-1}J(\mathbf{f})]$$

$$= \quad -D[J(\mathbf{f})^{\top}]\Gamma J(D^k\mathbf{x})J(D^{k-1}\mathbf{x})^{-1}J(\mathbf{f}) - J(\mathbf{f})^{\top}\Gamma D[J(D^k\mathbf{x})]J(D^{k-1}\mathbf{x})^{-1}J(\mathbf{f})$$

$$-J(\mathbf{f})^{\top}\Gamma J(D^k\mathbf{x})D[J(D^{k-1}\mathbf{x})^{-1}J(\mathbf{f})]$$

$$= \quad -D[J(\mathbf{f})^{\top}]\Gamma J(D^k\mathbf{x})J(D^{k-1}\mathbf{x})^{-1}J(\mathbf{f}) - J(\mathbf{f})^{\top}\Gamma D[J(D^k\mathbf{x})]J(D^{k-1}\mathbf{x})^{-1}J(\mathbf{f})$$

$$+J(\mathbf{f})^{\top}\Gamma J(D^k\mathbf{x})J(D^{k-1}\mathbf{x})^{-1}J(D^k\mathbf{x})J(D^{k-1}\mathbf{x})^{-1}J(\mathbf{f})$$

by Lemma 3.4 (iv). Multiply $J(\mathbf{f})^{-1}J(D^{k-1}\mathbf{x})J(D^k\mathbf{x})^{-1}J(\mathbf{f})$ on the right side and we get

$$0 = -D[J(\mathbf{f})^{\top}]\Gamma J(\mathbf{f}) - J(\mathbf{f})^{\top}\Gamma D[J(D^k\mathbf{x})]J(D^k\mathbf{x})^{-1}J(\mathbf{f}) - B^{(k)}.$$

Since

$$D[J(\mathbf{f})^\top] = -J(\mathbf{f})^\top J(D\mathbf{x})^\top \text{ and } D[J(D^k\mathbf{x})] = J(D^{k+1}\mathbf{x}) - J(D\mathbf{x})J(D^k\mathbf{x})$$

by Lemma 3.4 (iii) and (i), we finally have

$$0 = -(B^{(1)})^\top + B^{(k+1)} - B^{(1)} - B^{(k)}.$$

$\square$

Lemma 3.6. *Let k be a nonnegative integer. Suppose that $(E)_0$, $(E)_1, \ldots, (E)_k$ are all true. Then*
(i) $B^{(k+1)} = (k+1)B^{(1)} + k(B^{(1)})^\top$,
(ii) $\det B^{(k+1)}$ is a nonzero constant.

Proof. Using Lemma 3.5 repeatedly we have (i).

Write $B = B^{(1)}$ for simplicity. The shape of B is known [SoT, Lemma 3.9] as follows:

Case 1): If G is not of type D_l with l even then

(16)
$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & B_{1l} \\ 0 & 0 & \cdots & B_{2,l-1} & B_{2l} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & B_{l-1,2} & & B_{l-1,l-1} & B_{l-1,l} \\ B_{l1} & B_{l2} & \cdots & B_{l,l-1} & B_{ll} \end{pmatrix}$$

where

(17)
$$\begin{array}{ll} B_{ij} = 0 & \text{if } i + j < l + 1 \\ B_{ij} \in \mathbf{R}^* & \text{if } i + j = l + 1 \\ m_i B_{ij} = m_j B_{ji} & \text{if } i + j = l + 1 \,. \end{array}$$

Case 2): If G is of type D_l with $l = 2p$ then the 2×2 block in rows and columns $p, p+1$ of the matrix (16) - the center of the matrix - is to be replaced by a 2×2 block

$$B_0 = \begin{pmatrix} B_{p,p} & B_{p,p+1} \\ B_{p+1,p} & B_{p+1,p+1} \end{pmatrix}$$

with constant entries, where $B_{p,p+1} = B_{p+1,p}$ and $\det B_0 \in \mathbf{R}^*$. The statement (17) still holds true outside the 2×2 block B_0.

Therefore, by (i), we can verify the following statements:

Case 1): If G is not of type D_l with l even then

$$\begin{array}{ll} B_{ij}^{(k+1)} = 0 & \text{if } i + j < l + 1 \\ B_{ij}^{(k+1)} = (k+1)B_{ij} + kB_{ji} = \left(k + 1 + \frac{km_i}{m_j}\right)B_{ij} \in \mathbb{R}^* & \text{if } i + j = l + 1 \,. \end{array}$$

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Case 2): If G is of type D_l with $l = 2p$ then the 2×2 block $B_0^{(k+1)}$ in rows and columns $p, p+1$ of the matrix $B^{(k+1)}$ is

$$B_0^{(k+1)} = \begin{pmatrix} B_{p,p}^{(k+1)} & B_{p,p+1}^{(k+1)} \\ B_{p+1,p}^{(k+1)} & B_{p+1,p+1}^{(k+1)} \end{pmatrix} = (2k+1)B_0$$

with constant entries, where $\det B_0^{(k+1)} \in \mathbb{R}^*$. The statement (17) holds true outside $B_0^{(k+1)}$.

In both cases we conclude $\det B^{(k+1)} \in \mathbb{R}^*$. $\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 3.7. (i) From what we know about the shape of $B^{(k)}$ in the proof of Lemma 3.6, we know that the symmetric matrix $B^{(k)} + (B^{(k)})^\top$ has also nonzero constant determinant for each positive integer k. Since

$$B^{(1)} + (B^{(1)})^\top = J(\mathbf{f})^\top \Gamma D[J(\mathbf{f})] + D[J(\mathbf{f})]^\top \Gamma J(\mathbf{f}) = D[J(\mathbf{f})^\top \Gamma J(\mathbf{f})],$$

its (i, j)-entry is equal to the Yukawa coupling [Sai5, 5.3] of df_i and df_j. Therefore $B^{(1)}$ can be considered as an asymmetric "half" of the Yukawa coupling which is symmetric. The matrices $B^{(k)}$ are considered to be higher versions of $B^{(1)}$ in the sense that we iteratedly use the primitive derivation D to define them.

(ii) Basic invariants $f_1, \ldots, f_\ell$ are said to be flat generators when the matrix $B^{(1)} + (B^{(1)})^\top = D[J(\mathbf{f})^\top \Gamma J(\mathbf{f})]$ is a constant matrix. Flat generators always exist [Sai2]. They give a linear structure on the quotient variety V/G, which is a Frobenius manifold structure [Man, III. §8].

Proof of Theorems 1.1 and 1.2. Thanks to Lemma 3.2 it is enough to verify $(E)_k$ for $k \geq 0$. We prove $(E)_k$ by induction on k. The assertion $(E)_0$ is trivially true because
$$\mathsf{D}^{(0)}(\mathcal{A}) = \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in S\} = \mathrm{Der}_S$$
and $P_0 = \Gamma$. In this case $\xi_j(x_i) = (x_j, x_i)$ and $\deg \xi_j = 0$ for $1 \leq j \leq \ell$. Suppose that $(E)_0, (E)_1, \ldots, (E)_k$ are all true. We will show $(E)_{k+1}$. By Lemma 3.6, $\det B^{(k+1)}$ is a nonzero constant. Since

$$\begin{aligned} B^{(k+1)} &= -J(\mathbf{f})^\top \Gamma J(D^{k+1}\mathbf{x}) J(D^k\mathbf{x})^{-1} J(\mathbf{f}), \\ \det(J(D^k\mathbf{x})) &\doteq Q^{-2k}, \\ \det(J(\mathbf{f})) &\doteq Q, \end{aligned}$$

we obtain

$$\det(J(D^{k+1}\mathbf{x})) \doteq Q^{-2k-2}.$$

Let $P_{2k+2} = \Gamma J(D^{k+1}\mathbf{x})^{-1}$ and $(\xi_1, \ldots, \xi_\ell) = (\partial_1, \ldots, \partial_\ell)P_{2k+2}$. Then

$$\deg \xi_j = -\deg(\partial_i D^{k+1} x_j) = (k+1)h$$

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for $j = 1, \ldots, \ell$ by (8). Hence, in order to verify $(E)_{k+1}$, it suffices to show that each ξ_j belongs to $\mathsf{D}^{2k+2}(\mathcal{A})$ thanks to Ziegler's criterion 3.1. First we will show how $\xi_1, \ldots, \xi_\ell$ are affected by choosing a different basis for V. Suppose that a new basis $x'_1, \ldots, x'_\ell$ for V is connected to the old basis $x_1, \ldots, x_\ell$ through an invertible matrix $M \in GL_\ell(\mathbb{R})$ as (10). Then, by (11), (12), and (14), we obtain

$$
\begin{aligned}
(\xi'_1, \ldots, \xi'_\ell) &= (\partial'_1, \ldots, \partial'_\ell) \Gamma' J'(D^{k+1}\mathbf{x}')^{-1} \\
&= (\partial_1, \ldots, \partial_\ell)(M^\top)^{-1} M^\top \Gamma M M^{-1} J(D^{k+1}\mathbf{x})^{-1} M \\
&= (\xi_1, \ldots, \xi_\ell) M.
\end{aligned}
$$

In other words, $\xi_1, \ldots, \xi_\ell$ satisfy the same base change rule as $\mathbf{x} = (x_1, \ldots, x_\ell)$.

Let $H \in \mathcal{A}$. Then we may assume that $x_1 = \alpha_H, x_2, \ldots, x_\ell$ are an orthonormal basis. It is enough to show that $\xi_j(x_1) \in S x_1^{2k+2}$ for each j. Since $(O)_k$ follows from $(E)_k$ by Lemma 3.2, each entry of the first row of $J(D^{k+1}\mathbf{x})^{-1} J(\mathbf{f})$ is divisible by x_1^{2k+1}. By the exactly same argument as in the proof of Lemma 3.3, we know that each entry of $J(D^{k+1}\mathbf{x})$, except the first column, has no pole along $x_1 = 0$. Thus each entry of the first row of $\mathrm{adj}\, J(D^{k+1}\mathbf{x})$ has no pole along $x_1 = 0$. Since $\det J(D^{k+1}\mathbf{x}) \doteq Q^{-2k-2}$, each entry of the first row of $J(D^{k+1}\mathbf{x})^{-1}$ is divisible by x_1^{2k+2}. This implies $\xi_j(x_1) \in S x_1^{2k+2}$ for each j. Thus $(E)_{k+1}$ holds true and the induction proceeds. This completes the proof of Theorems 1.1 and 1.2.

Example 3.8. (The case B_2) Let us explicitly calculate $J(D\mathbf{x}), P_3, \xi_1^{(3)}, \xi_2^{(3)}$ when $\mathcal{A}$ is of type B_2. Let x_1 and x_2 be an orthonormal basis and

$$
f_1 = \frac{1}{2}(x_1^2 + x_2^2), \quad f_2 = \frac{1}{4}(x_1^4 + x_2^4).
$$

Then

$$
P_1 = J(\mathbf{f}) = \begin{pmatrix} x_1 & x_1^3 \\ x_2 & x_2^3 \end{pmatrix}.
$$

Choose $Q = \det J(\mathbf{f}) = x_1 x_2 (x_1 + x_2)(x_2 - x_1)$. Since

$$
J(\mathbf{f})^{-1} = \frac{1}{Q} \begin{pmatrix} x_2^3 & -x_1^3 \\ -x_2 & x_1 \end{pmatrix},
$$

one has

$$
D = -\frac{x_2}{Q}\partial_1 + \frac{x_1}{Q}\partial_2, \quad D\mathbf{x} = -\left(\frac{x_2}{Q}, \frac{x_1}{Q}\right),
$$

$$
J(D\mathbf{x}) = \frac{1}{Q^2} \begin{pmatrix} (3x_1^2 - x_2^2)x_2^2 & 2x_1^3 x_2 \\ 2x_1 x_2^3 & x_1^2(x_1^2 - 3x_2^2) \end{pmatrix},
$$

$$
\begin{aligned}
P_3 &= J(D\mathbf{x})^{-1} J(\mathbf{f}) \\
&= \begin{pmatrix} -(1/3)x_1^3(x_1^2 - 5x_2^2) & -(1/3)x_1^3(x_1^4 - 3x_1^2 x_2^2 - 2x_2^4) \\ (1/3)x_2^3(5x_1^2 - x_2^2) & (1/3)x_2^3(2x_1^4 + 3x_1^2 x_2^2 - x_2^4) \end{pmatrix}.
\end{aligned}
$$

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Thus the basis $\xi_1^{(3)}, \xi_2^{(3)}$ for $D^{(3)}(\mathcal{A})$ are given by

$$
\begin{aligned}
\xi_1^{(3)} &= -(1/3)x_1^3(x_1^2 - 5x_2^2)\partial_1 + (1/3)x_2^3(5x_1^2 - x_2^2)\partial_2, \\
\xi_2^{(3)} &= -(1/3)x_1^3(x_1^4 - 3x_1^2x_2^2 - 2x_2^4)\partial_1 + (1/3)x_2^3(2x_1^4 + 3x_1^2x_2^2 - x_2^4)\partial_2.
\end{aligned}
$$

We have the following alternative inductive expression for the matrices P_m in Theorem 1.1 using the matrices $B^{(k)}$:

Proposition 3.9. *For any nonnegative integer m, we have*

$$
P_m = \begin{cases}
\Gamma & (\text{if } m = 0), \\
P_{m-1}J(\mathbf{f}) & (\text{if } m \text{ is odd}), \\
-P_{m-1}(B^{(m/2)})^{-1}P_1^\top & (\text{if } m \text{ is positive and even}).
\end{cases}
$$

Proof. Only the third formula needs to be verified. Let $m = 2k$. Compute

$$
\begin{aligned}
& -P_{2k-1}(B^{(k)})^{-1}P_1^\top \\
=\ & -\Gamma J(D^{k-1}\mathbf{x})^{-1}J(\mathbf{f})(-J(\mathbf{f})^\top \Gamma J(D^k\mathbf{x})J(D^{k-1}\mathbf{x})^{-1}J(\mathbf{f}))^{-1}J(\mathbf{f})^\top \Gamma \\
=\ & \Gamma J(D^k\mathbf{x})^{-1} = P_{2k}.
\end{aligned}
$$

$\square$

Example 3.10. (The case B_ℓ) Let G be the Coxeter group of type B_ℓ acting on an ℓ-dimensional Euclidean space V by signed permutations of an orthonormal basis $e_1, \ldots, e_\ell$. Let $x_1, \ldots, x_\ell$ be the dual basis for V^*. Then Γ is the identity matrix. Define

$$
f_i = f_i(x_1, \ldots, x_\ell) = \frac{1}{2i}\sum_{j=1}^{\ell} x_j^{2i}
$$

for $1 \le i \le \ell$. We will use the basic invariants $f_1, \ldots, f_\ell$. Then

$$
P_1 = J(\mathbf{f}) = \begin{pmatrix} x_1 & x_1^3 & \cdots & x_1^{2\ell-1} \\ \vdots & \vdots & & \vdots \\ x_\ell & x_\ell^3 & \cdots & x_\ell^{2\ell-1} \end{pmatrix}.
$$

For $i \ge 0$ let

$$
h_i(x_1, \ldots, x_\ell) = \sum_{i_1 + \cdots + i_\ell = i} x_1^{i_1} \cdots x_\ell^{i_\ell}
$$

be the i-th complete symmetric polynomial. Let

$$
\tilde{h}_i = \tilde{h}_i(x_1, \ldots, x_\ell) = h_i(x_1^2, \ldots, x_\ell^2).
$$

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Then $\tilde{h}_i$ is a G-invariant polynomial of degree $2i$. Define $\tilde{h}_i(x_1, \ldots, x_\ell) = 0$ if $i < 0$. In [SoT, §5.2], $B^{(1)}$ was calculated as:

$$B^{(1)}_{ij} = (2j - 1)\tilde{h}_{i+j-\ell-1}(x_1, \ldots, x_\ell)$$

for $1 \leq i, j \leq \ell$. Then, by Lemma 3.6, we have

$$B^{(k)}_{ij} = kB^{(1)}_{ij} + (k - 1)B^{(1)}_{ji} = \{k(2i + 2j - 2) - 2i + 1\}\tilde{h}_{i+j-\ell-1}(x_1, \ldots, x_\ell)$$

for $1 \leq i, j \leq \ell$ and $k \geq 1$. We can inductively calculate P_m using Proposition 3.9.

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